

# Alaa Eldin Tamer

## Sec 13

**Matrix algebra:** If  $m, n$  are positive integers then an  $m \times n$  matrix is a rectangular array

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad m \times n = \text{size of matrix}$$

$\downarrow \quad \downarrow$   
 rows column

**Special case:**  $m=n$  (square matrix)

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 0 & 3 \\ -2 & -3 & 4 \end{pmatrix} \quad \text{The main diagonal}$$

**Equality of matrices:** Two matrices  $A(a_{ij})$  and  $B(b_{ij})$  are equal if they have the same size ( $m \times n$ ) and  $a_{ij} = b_{ij}$

$$\text{Ex) } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$$

$2 \times 2 \quad \quad 2 \times 1 \quad \quad 1 \times 2 \quad \quad 2 \times 2$

- if  $x=4$  then  $A$  and  $D$  are equal, otherwise they are not equal
- $B \neq C$  because they don't have same size

Matrix of one row or one column

$$A = (3 \ 2 \ 5 \ 8 \ 0) \quad \text{row matrix (row vector)}$$

$$B = \begin{pmatrix} 8 \\ 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad \text{column matrix (column vector)}$$



## Matrix operations

### 1. Matrix addition

Two matrices are added only if they have same size

ex

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \end{pmatrix}$$

$$\begin{matrix} 2 \times 2 & 2 \times 2 \end{matrix} \quad A+B = \begin{pmatrix} 0 & 5 \\ -1 & 3 \end{pmatrix} \quad \begin{matrix} 2 \times 2 & 1 \times 2 \end{matrix} \quad A+C = \text{undefined}$$

### 2. Scalar multiplication

When multiplying a matrix with a number, we multiply all entries in that matrix

ex

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix} \quad 3A = \begin{pmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 9 \end{pmatrix}$$

### 3. Matrix multiplication

$$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \quad \begin{matrix} A & \times & B \\ 3 \times 2 & & 2 \times 2 \\ \text{equal} \end{matrix} = C \quad \begin{matrix} 3 \times 2 \end{matrix}$$

For two matrices to be multiplied, the no. of columns in first matrix is the same as no. of rows in the second matrix

$A \times B$  can be multiplied  $3 \times 2 \times 2 \times 2$

$B \times A$  can't be multiplied  $2 \times 2 \times 3 \times 2$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{pmatrix}$$

To get  $C_{11}$  multiply row 1 in A by column 1 in B

To get  $C_{31}$  multiply row 3 in A by column 1 in B

To get  $C_{22}$  multiply row 2 in A by column 2 in B

\* The product matrix has no. of rows of first matrix and no. of columns of the second matrix



$$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix} \times B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \Rightarrow C = \begin{pmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{pmatrix}$$

$$C_{11} = (-1 \times -3) + (3 \times 4) = 9 \quad C_{21} = (4 \times -3) + (-2 \times -4) = -4$$

$$C_{12} = (-1 \times 2) + (3 \times 1) = 1 \quad C_{22} = (4 \times 2) + (-2 \times 1) = 6$$

$$Ex. \quad A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

$$A \times B = C \quad \begin{pmatrix} 2 & 5 \\ 4 & -4 \end{pmatrix}$$

$2 \times 2 \quad 2 \times 2 \quad 2 \times 2$

$$Ex. \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 5 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 11 \end{pmatrix}$$

$3 \times 3 \quad 3 \times 1 \quad 3 \times 1$

$$Ex. \quad \begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 \\ 0 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -3 & 2 \end{pmatrix}$$

$2 \times 4 \quad 4 \times 2 \quad 2 \times 2$

## Properties of matrix operations

- 1-  $A+B = B+A$
- 2-  $AB \neq BA$  in general
- 3-  $A+(B+C) = (A+B)+C$
- 4-  $C(A+B) = CA+CB$
- 5-  $A(BC) = (AB)C \neq (AC)B$
- 6-  $A(B+C) = AB+AC$



Note:  $(A+B)(A-B) \neq A^2 - B^2$

Ex.  $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$   $B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$  Find  $AB, BA$

$$AB = \begin{pmatrix} 2 & 5 \\ 4 & -4 \end{pmatrix} \quad BA = \begin{pmatrix} 0 & 7 \\ 4 & -2 \end{pmatrix} \quad AB \neq BA$$

Ex.  $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$   $B = \begin{pmatrix} -2 & 4 \\ 2 & -2 \end{pmatrix}$  find  $AB, BA$

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad BA = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad AB = BA$$

~~Special and rare~~  
Case

If  $AC = BC$ ,  ~~$A \neq B$~~  then  $A$  not necessarily equals  $B$

Ex.  $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$   $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$   $C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$

$$AC = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \quad BC = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$$

$$AC = BC \text{ but } A \neq B$$

### Identity matrix

The identity matrix of order  $(n)$  is a square matrix that has 1's on the main diagonal and 0's elsewhere

Ex.  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$   $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



\* When we multiply any matrix by identity matrix, the matrix does not change

$$\text{eg. } \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$$

The transpose of a matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -3 & -2 & 4 \end{pmatrix}_{2 \times 4} \quad A^T = \begin{pmatrix} 2 & 0 \\ 1 & -3 \\ 3 & -2 \\ 5 & 4 \end{pmatrix}_{4 \times 2}$$

Properties

1.  $(A^T)^T = A$

2.  $(A+B)^T = A^T + B^T$

3.  $(AB)^T = B^T A^T$

4. if  $A = A^T \Rightarrow$  Symmetric matrix

if  $A = -A^T \Rightarrow$  Skew-Symmetric matrix